

## On the complete Seiberg-Witten map for theories with topological terms

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**ABSTRACT:** The SW map problem is formulated and solved in the BRST cohomological approach. The well known ambiguities of the SW map are shown to be associated to distinct cohomological classes. This analysis is applied to the noncommutative Chern-Simons action resulting in the emergence of  $\theta$ -dependent terms in the commutative action which come from the nontrivial ambiguities. It is also shown how a specific cohomological class can be chosen in order to map the noncommutative Maxwell-Chern-Simons theory into the commutative one.

**KEYWORDS:** Non-Commutative Geometry, BRST Symmetry, Nonperturbative Effects.

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**1. Introduction**

Distinct gauge choices in the open strings lead both to the realization of ordinary Yang-Mills field theories as well as to noncommutative field theories. It was the perception of this fact that made Seiberg and Witten propose what became known as the Seiberg-Witten map (SW map) [1]. In brief words, this map establishes a transformation of noncommutative field variables in terms of ordinary (commutative) fields, in such a way that the noncommutative gauge transformation is mapped into the ordinary one.

Let us express this mathematically. First we introduce the Moyal product between two functions defined on the noncommutative space [2]:

$$\begin{aligned}
 f * g &= \exp\left(\frac{i}{2}\theta^{ij}\frac{\partial}{\partial x^i}\frac{\partial}{\partial y^j}\right) f(x)g(y)_{y\rightarrow x} \\
 &= fg + \frac{i}{2}\theta^{ij}\partial_i f \partial_j g + O(\theta^2),
 \end{aligned}
 \tag{1.1}$$

where the real c-number parameter  $\theta^{ij}$  comes from the noncommutativity of the space-time coordinates

$$[x^i, x^j] = i\theta^{ij}.
 \tag{1.2}$$

Making  $\theta^{ij}$  vanish brings the noncommutative theory into the commutative one. Then, after defining the Moyal bracket,

$$[f * , g] = f * g - g * f,
 \tag{1.3}$$

we are able to construct a noncommutative gauge transformation,

$$\hat{\delta}_\lambda^\wedge A = \partial_i \hat{\lambda} + i \left[ \hat{\lambda} * , \hat{A}_i \right] = \hat{D}_i \hat{\lambda},
 \tag{1.4}$$

where the hat symbol identifies fields and operators defined on the noncommutative space-time and  $\hat{D}_i$  represents the Moyal covariant derivative. At this point it should be observed that, although similar in form to a nonabelian gauge transformation, a nonvanishing contribution coming from the Moyal bracket in (1.4) is expected even for an abelian gauge field. In the same way, the noncommutative gauge curvature

$$\hat{F}_{ij} = \partial_i \hat{A}_j - \partial_j \hat{A}_i - i \left[ \hat{A}_i, \hat{A}_j \right] \tag{1.5}$$

gets a nonvanishing contribution coming from the commutator even for an abelian field. In the abelian case, up to first order in  $\theta^{ij}$ , we get

$$\delta_\lambda \hat{A} = \partial_i \hat{\lambda} - \theta^{kl} \partial_k \hat{\lambda} \partial_l \hat{A}_i + O(\theta^2), \tag{1.6}$$

$$\hat{F}_{ij} = \partial_i \hat{A}_j - \partial_j \hat{A}_i + \theta^{kl} \partial_k \hat{A}_i \partial_l \hat{A}_j + O(\theta^2). \tag{1.7}$$

Now, the sense of the SW map is that noncommutative gauge equivalent fields should be mapped into ordinary gauge equivalent fields. If we express this map as a formal series  $\hat{A}_i(A)$ , what we are saying is that

$$\delta_\lambda \left[ \hat{A}_i(A) \right] = \left[ \delta_\lambda \hat{A}_i \right] (A), \tag{1.8}$$

where  $\delta_\lambda$  is the ordinary gauge operation acting on  $A_i$ . From the abelian expression (1.6), we see that the SW map cannot be a simple field redefinition  $\hat{A}_i = \hat{A}_i(A, \theta)$  together with a reparametrization of  $\hat{\lambda} = \hat{\lambda}(\lambda, \theta)$ . The  $U(1)$  case is symptomatic: such choice of redefinitions would never get rid of the Moyal term in (1.6), which comes from the noncommutative nature of space-time, in contrast with the absence of noncommutative terms in an ordinary  $U(1)$  gauge transformation. The only hope is to mix  $\lambda$  and  $A$  in the SW map of  $\hat{\lambda}$ :  $\hat{\lambda} = \hat{\lambda}(\lambda, A, \theta)$ .

In first order in  $\theta$ , we can write [1]

$$\hat{A}_i(A) = A_i - \theta^{kl} \left( A_k \partial_l A_i - \frac{1}{2} A_k \partial_i A_l \right) + O(\theta^2), \tag{1.9}$$

$$\hat{\lambda}(\lambda, A) = \lambda + \frac{1}{2} \theta^{kl} (\partial_k \lambda) A_l + O(\theta^2). \tag{1.10}$$

One can also establish how  $\hat{A}_i$  and  $\hat{\lambda}$  would change if we allow for variations in the  $\theta^{ij}$  parameter. This problem is, in fact, analogous to that of the SW map in first order, as it is its solution [1]:

$$\delta_\theta \hat{A}_i = -\frac{1}{4} \delta \theta^{kl} \left\{ \hat{A}_k, \partial_l \hat{A}_i + \hat{F}_{li} \right\}, \tag{1.11}$$

$$\delta_\theta \hat{\lambda} = \frac{1}{4} \delta \theta^{kl} \left\{ \partial_k \hat{\lambda}, \hat{A}_l \right\}. \tag{1.12}$$

Equations (1.11) and (1.12) represent a system of coupled differential equations. But, as we will discuss in sections 3 and 4, this system is not the most general one compatible with the SW condition (1.8). And, from this point of view, equations (1.9) and (1.10) only represent a first order particular solution. This is the source for the ambiguity of the SW map already observed in the literature [1, 3]. It also should be stressed that eq. (1.11) has been the starting point for an interesting result, when it was argued that the SW map would transform the 3D noncommutative Chern-Simons theory into the ordinary Chern-Simons theory with no corrections in  $\theta^{ij}$  whatsoever. This first exact result in the literature for the SW map was obtained by just showing that [4]

$$\delta_\theta \hat{S}_{NCCS} = 0 \tag{1.13}$$

where  $\hat{S}_{NCCS}$  stands for the 3D noncommutative Chern-Simons (*NCCS*) action and  $\delta_\theta$  is the operation defined in eq. (1.11).

This result surprises as it is not obvious that the substitution of equation (1.9) on the *NCCS* action would promote an exact cancellation of the  $\theta$  terms order by order in  $\theta$ , in such a way that all the noncommutative contributions coming from the Moyal products be gone, leaving just the usual commutative Chern-Simons action,

$$\hat{S}_{NCCS} \xrightarrow{SW\ map} S_{CS}. \tag{1.14}$$

As a comparison, no such result has been found up to now for any other theory. Then, the questions are if one can find other examples of such perfect mappings and what would be the property for a theory to allow these mappings. Answering these questions is the aim of this paper.

So, we begin the work in the next section by exploring the ambiguity in the solution of (1.11) and (1.12) from a different point of view, coming from the cohomology of the operators in play. We will derive the complete solution of the cohomology problem associated to the SW map. This will clarify the question if these ambiguities would damage the exact SW map of *NCCS*. We will also briefly discuss the integrability of equation (1.11) in higher orders in  $\theta$ . With these developments, we will be able to generalize this equation and show that also a theory as noncommutative Chern-Simons plus Maxwell (*NCMCS*) in 3D can be exactly mapped into the ordinary *MCS* theory avoiding any  $\theta$  corrections. Our final conclusion claims for a conjecture that such exact mappings will always be possible in the presence of topological terms, but also that  $\theta$  corrections will be unavoidable when such terms are absent in pure geometrical theories. Some quantum aspects can then be expected from this conjecture.

## 2. Ambiguities

From this point on, we will restrict ourselves to the abelian U(1) case. The question of the ambiguities on the SW map can be traced from the disclosure of the map itself. Soon it was noticed [1, 3] that the solution of the SW map (1.9) was not uniquely defined. This can be foreseen from the definition of the map. It only requests that noncommutative gauge

equivalent classes should be mapped into commutative gauge equivalent classes, so leading to a commutative gauge invariant theory in the end. But this theory will remain gauge invariant if we allow for a subsequent transformation in the commutative gauge field as long as it has the form of a (field dependent) gauge transformation or a gauge invariant field redefinition [1, 3]. So, if we make a new transformation made by the composition of the original SW map (1.9) and the gauge transformation or field redefinition of the commutative connection, we will end with a map which will be again a SW map.

Some questions naturally arise in this context. The first is if there is space for other ambiguities than those that we have listed above. The second is what would be the (mathematical and physical) consequences of these ambiguities in the SW mapping of a given noncommutative theory.

We will approach these questions from the point of view of BRST cohomological techniques [5–7]. In this language, the gauge transformation of the commutative gauge field is described by the action of the nilpotent BRST differential  $s$ ,

$$sA_i = \partial_i c, \quad sc = 0, \quad s^2 = 0, \tag{2.1}$$

where  $c$  is the commutative ghost field. As before, we take  $\hat{C}$  and  $\hat{A}_i$ , the noncommutative fields, as formal power series in  $\theta$  whose coefficients are local polynomials in  $c$  and  $A_i$  (and in their derivatives). Then equation (1.8) for the SW map is translated as [5]

$$s \hat{A}_i = \partial_i \hat{C} + i \left[ \hat{C}; \hat{A}_i \right], \tag{2.2}$$

and

$$s \hat{C} = i \hat{C} * \hat{C}. \tag{2.3}$$

We write the series in  $\theta$  as

$$\hat{A}_i = A_i + \sum_{n=1}^{\infty} A_i^{(n)}, \tag{2.4}$$

$$\hat{C} = c + \sum_{n=1}^{\infty} C^{(n)}, \tag{2.5}$$

where  $A_i^{(n)}$  and  $C^{(n)}$  identify the term with  $n$   $\theta$ 's in the power series expansion of the fields. Now we are able to expand equations (2.2) and (2.3) as well. In their  $n$ 'th order, we find

$$sA_i^{(n)} = \partial_i C^{(n)} - \sum_{\alpha=1}^n \theta^{kl} \partial_k C^{(n-\alpha)} \partial_l A_i^{(\alpha-1)} + \dots \tag{2.6}$$

$$+ \frac{i^{n+1}}{2^n n!} \theta^{k_1 l_1} \dots \theta^{k_n l_n} (\partial_{k_1} \dots \partial_{k_n} c \partial_{l_1} \dots \partial_{l_n} A_i - \partial_{k_1} \dots \partial_{k_n} A_i \partial_{l_1} \dots \partial_{l_n} c),$$

$$sC^{(n)} = -\frac{1}{2} \sum_{\alpha=1}^n \theta^{kl} \partial_k C^{(n-\alpha)} \partial_l C^{(\alpha-1)} + \dots \tag{2.7}$$

$$+ \frac{i^{n+1}}{2^n n!} \theta^{k_1 l_1} \dots \theta^{k_n l_n} \partial_{k_1} \dots \partial_{k_n} c \partial_{l_1} \dots \partial_{l_n} c,$$

with  $A_i^{(0)} = A_i$  and  $C^{(0)} = c$ . Then, up to second order in  $\theta$ ,

$$sA_i = \partial_i c, \tag{2.8}$$

$$sA_i^{(1)} = \partial_i C^{(1)} - \theta^{kl} \partial_k c \partial_l A_i, \tag{2.9}$$

$$sA_i^{(2)} = \partial_i C^{(2)} - \theta^{kl} \left( \partial_k C^{(1)} \partial_l A_i + \partial_k c \partial_l A_i^{(1)} \right), \tag{2.10}$$

and

$$sc = 0, \tag{2.11}$$

$$sC^{(1)} = -\frac{1}{2} \theta^{kl} \partial_k c \partial_l c, \tag{2.12}$$

$$sC^{(2)} = -\theta^{kl} \partial_k c \partial_l C^{(1)}. \tag{2.13}$$

$$\tag{2.14}$$

The first order in  $\theta$  equation, where the map begins, has a solution which can be divided into two parts. The first part is any particular solution of the “inhomogeneous” equations, where the “inhomogeneity” is characterized by the explicit  $\theta$ -dependent term in (2.9) and (2.12). Let us call this part as  $a_i^{(1)}$  and  $c^{(1)}$  respectively. The second part is the general solution  $(\mathbb{A}^{(1)}, \mathbb{C}^{(1)})$  for the associated homogeneous equations

$$s\mathbb{A}_i^{(1)} = \partial_i \mathbb{C}^{(1)}, \tag{2.15}$$

and

$$s\mathbb{C}^{(1)} = 0. \tag{2.16}$$

The first part of the solution,  $(a_i^{(1)}, c^{(1)})$ , became known as the first order SW map [1], which can be read from eqs. (1.9) and (1.10)

$$a_i^{(1)}(A) = -\theta^{kl} \left( A_k \partial_l A_i - \frac{1}{2} A_k \partial_i A_l \right), \tag{2.17}$$

$$c^{(1)}(c, A) = \frac{1}{2} \theta^{kl} (\partial_k c) A_l.$$

Now, the general solution  $\mathbb{A}_i^{(1)}$  is the source of all the ambiguities and freedom in the SW map that the literature refers to [1, 3]. Eq. (2.15), together with the nilpotency of  $s$  assured in (2.1), happens to be a problem of the cohomology of the BRST operator modulo total derivatives (a review of the cohomology of BRST can be found in [8]).  $\mathbb{A}_i^{(1)}$  is the most general polynomial satisfying (2.15) constructed with the commutative gauge field, derivatives and one  $\theta$ , in the sector with ghost number zero, canonical dimension 1 and carrying a free Lorentz index (remembering eqs. (1.1) and (1.2), we can associate a dimension  $-2$  to  $\theta$ ). Eq. (2.16), which is a direct consequence of the nilpotency of  $s$  and of eq. (2.15), is a problem of the local cohomology of  $s$ , in the sector with ghost number 1 and zero canonical dimension.

In fact, the set of eqs. (2.15) and (2.16) will be found in each sector  $n$  of the  $\theta$  expansion of (2.2) and (2.3)

$$s\mathbb{A}_i^{(n)} = \partial_i \mathbb{C}^{(n)}, \tag{2.18}$$

and

$$s\mathbb{C}^{(n)} = 0. \quad (2.19)$$

The solutions will change as the required number of  $\theta$ 's change for each  $n$ .

In the study of the cohomology of BRST modulo total derivatives ( $\mathcal{H}(s/d)$ ) one can find an analogous set of equations as this above. They are known as descent equations, and they appear in the analysis of the quantum stability and anomalies of gauge theories [8]. In the seek of completion, we will briefly review the general ideas of this study. Let us write generic descent equations with the same Lorentz structure as (2.18) and (2.19)

$$sW_i = \partial_i X, \quad (2.20)$$

and

$$sX = 0, \quad (2.21)$$

where  $W_i$  and  $X$  are formal power series in the field and parameter space of the gauge theory. We have to solve this problem beginning by the last equation of the set, (2.21), which is a simpler problem of the local cohomology. The solutions of (2.21) are classified as trivial, those which are written as simple BRST variations of polynomials in the field space  $P(\varphi)$  of the problem, or non-trivial solutions, invariant polynomials which cannot be written in this way,

$$X = Y + sZ, \quad | sY = 0 \text{ and } Y \neq s\Omega, \forall \Omega \in P(\varphi). \quad (2.22)$$

We say that  $Y$  belongs to the local cohomology of the  $s$  operator,  $Y \in \mathcal{H}(s)$ . We can act with a derivative on (2.22) and substitute the result on (2.20),

$$s(W_i - \partial_i Z) = \partial_i Y. \quad (2.23)$$

Let us say that we can find a particular solution to this equation, let us call it  $\delta(\partial_i Y)$ , such that

$$s\delta(\partial_i Y) = \partial_i Y. \quad (2.24)$$

It is important to observe that sometimes it is not possible to find such particular solutions coming from non-trivial contributions of lower level descent equations [8]. When this happens, we say that this solution is obstructed and we are forced to make null its coefficient in order to continue the procedure. Substituting (2.24) on (2.23), we get

$$s(W_i - \delta(\partial_i Y) - \partial_i Z) = 0. \quad (2.25)$$

Now, this is again a problem of the local cohomology of  $s$ , and again the solutions will be classified as in (2.22)

$$W_i - \delta(\partial_i Y) - \partial_i Z = Y_i + sZ_i \quad | sY_i = 0 \text{ and } Y_i \neq s\Omega_i, \forall \Omega_i \in P(\varphi). \quad (2.26)$$

So, finally, the general solution of (2.20) is

$$W_i = Y_i + \delta(\partial_i Y) + sZ_i + \partial_i Z. \quad (2.27)$$

The terms given by  $Y_i + \delta(\partial_i Y)$  represent the nontrivial contributions to the cohomology of  $s/d$ , and  $sZ_i + \partial_i Z$  is the trivial part which is written as  $s$  or  $\partial_i$  variations of cocycles in the field space. In many problems of interest in quantum field theory, the trivial solutions are discarded for not carrying physical information. This happens in the study of the stability and anomalies in QFT. In what concerns the present problem of the SW map, this cohomological classification will also be of mathematical and physical relevance, as we will see in a moment.

We can return now to our specific problem. We want to solve the cohomological problem posed by equations (2.18) and (2.19) in the field space generated by  $A_i$ ,  $c$ , and derivatives acting on them, in the presence of  $\theta$  parameters (and obviously of all other parameters the theory under study allows),

$$P_{\text{SW}} = \{\theta, A_i, c, \partial_i\}. \tag{2.28}$$

We first argue that trivial  $s$  cocycles do not exist at the level of the upper descent equation (2.18) for any  $n$ . Notice that  $\mathbb{A}_i^{(n)}$  has zero ghost number, and as there is no element in the field space  $P_{\text{SW}}$  with negative ghost number, it is not possible to find any  $Z_i$  with ghost number  $-1$  as required by eq. (2.27). Next we will prove that neither there are contributions of the form  $\delta(\partial_i Y)$  at the upper level .

**Proposition 2.1** The only nontrivial solutions of the  $s/d$  cohomological problem given by eqs. (2.18) and (2.19) belong to the local  $s$  cohomology  $\mathcal{H}(s)$  at the upper level descent equation.

**Proof:** The local BRST cohomology of (abelian) gauge theories for arbitrary quantum numbers has already been extensively studied. It is given by polynomials built with the ghost  $c$  non derivated and the curvature  $F_{ij} = \partial_i A_j - \partial_j A_i$  possibly derivated [8]. In the case of eq. (2.19), we are looking for the cohomology of  $s$  in the sector of ghost number 1, zero dimension and in the presence of a  $n$  number of  $\theta$ 's. So, the form of the general element of  $\mathcal{H}(s)$  in this sector is

$$\alpha c \theta^{k_1 l_1} \dots \theta^{k_n l_n} . P_{k_1 l_1 \dots k_n l_n}(\partial_i, F_{ij}), \tag{2.29}$$

where  $\alpha$  is a numerical coefficient and  $P_{k_1 l_1 \dots k_n l_n}(\partial_i, F_{ij})$  represent all possible  $s$  invariant polynomials constructed with an arbitrary number of derivatives of the curvature, with free indices  $k_1 l_1 \dots k_n l_n$ , and total dimension  $2n$ . This is the most general nontrivial solution for  $\mathbb{C}^{(n)}$ . Following the steps from (2.22) to (2.24), in order to find the contribution in the upper level coming from this lower level descent equation nontrivial solution, we must solve

$$\begin{aligned} s\mathbb{A}_i^{(n)} &= \alpha \partial_i c \theta^{k_1 l_1} \dots \theta^{k_n l_n} . P_{k_1 l_1 \dots k_n l_n}(\partial, F) + \alpha c \theta^{k_1 l_1} \dots \theta^{k_n l_n} . \partial_i P_{k_1 l_1 \dots k_n l_n}(\partial, F) \\ &= s \left( \alpha A_i \theta^{k_1 l_1} \dots \theta^{k_n l_n} . P_{k_1 l_1 \dots k_n l_n}(\partial, F) \right) + \alpha c \theta^{k_1 l_1} \dots \theta^{k_n l_n} . \partial_i P_{k_1 l_1 \dots k_n l_n}(\partial, F). \end{aligned} \tag{2.30}$$

It becomes clear now that there is no possible solution for  $\mathbb{A}_i^{(n)}$  as the last term on (2.30) is again an element of  $\mathcal{H}(s)$  and in this way cannot be written as a  $s$  variation of any cocycle



of the field space. This characterizes the kind of obstruction we just mentioned, and the way out is to make  $\alpha = 0$ . Then, the only non-trivial solutions to (2.18) are those of the local cohomology  $\mathcal{H}(s)$  in the sector of ghost number zero, dimension equals to one, in the presence of a  $n$  number of  $\theta$ 's and with a free Lorentz index. **QED**

Let us call this local cohomology at the upper level equation (2.18) as  $\mathcal{H}^{(n)}(s)$ . The general element of  $\mathcal{H}^{(n)}(s)$  will be of the form

$$Y_i^{(n)} = \alpha \theta^{k_1 l_1} \dots \theta^{k_n l_n} \cdot P_{ik_1 l_1 \dots k_n l_n}(\partial, F), \quad (2.31)$$

where  $\alpha$  is a numerical coefficient and  $P_{ik_1 l_1 \dots k_n l_n}(\partial, F)$  represent all possible  $s$  invariant polynomials constructed with an arbitrary number of derivatives of the curvature, with free indices  $ik_1 l_1 \dots k_n l_n$ , and total dimension  $2n + 1$ . Finally, we find that the most general solutions to the eqs. (2.18) and (2.19) are given by

$$\mathbb{C}^{(n)} = sZ^{(n)}, \quad (2.32)$$

and

$$\mathbb{A}_i^{(n)} = Y_i^{(n)} + \partial_i Z^{(n)}, \quad (2.33)$$

where  $Z^{(n)}$  are generic polynomials with zero ghost number and dimension, built with elements of the field space  $P_{\text{SW}}$  in the presence of  $n$   $\theta$ 's. Notice that, unlike  $Y_i^{(n)}$ , the  $Z^{(n)}$  polynomials need not be invariant under the action of the BRST operator.

Having solved the cohomology associated to the problem of the SW map, we can now ask what changes the solutions  $\mathbb{A}_i^{(n)}$  can bring in the higher orders of the SW map once they are defined. For simplicity, let us say that we have found these cocycles in a given theory already at the first order,  $n = 1$ ,

$$\mathbb{C}^{(1)} = sZ^{(1)}, \quad (2.34)$$

$$\mathbb{A}_i^{(1)} = Y_i^{(1)} + \partial_i Z^{(1)}. \quad (2.35)$$

The second order equations for the SW map would be

$$sA_i^{(2)} = \partial_i C^{(2)} - \theta^{kl} \left( \partial_k \left( c^{(1)} + \mathbb{C}^{(1)} \right) \partial_l A_i + \partial_k c \partial_l \left( a_i^{(1)} + \mathbb{A}_i^{(1)} \right) \right), \quad (2.36)$$

$$sC^{(2)} = -\theta^{kl} \partial_k c \partial_l \left( c^{(1)} + \mathbb{C}^{(1)} \right), \quad (2.37)$$

where  $a_i^{(1)}$  and  $c^{(1)}$  are the particular solutions given in (2.17). We can make our usual division of the solution into the particular solution to the system above  $\left( a_i^{(2)}, c^{(2)} \right)$  and the general solution of the associated homogeneous system  $\left( \mathbb{A}_i^{(2)}, \mathbb{C}^{(2)} \right)$ . This latter system will be of the form of (2.18) and (2.19) for  $n = 2$ , and will be solved once we find the polynomials  $Y_i^{(2)}$  and  $Z^{(2)}$ , and substitute them on the general equations (2.32) and (2.33). The former system can be further split into a part that depends on the solutions of the cohomology of the first level, which we call  $\left( a_i^{R(2)}, c^{R(2)} \right)$ ,

$$s a_i^{R(2)} = \partial_i c^{R(2)} - \theta^{kl} \left( \partial_k \mathbb{C}^{(1)} \partial_l A_i + \partial_k c \partial_l \mathbb{A}_i^{(1)} \right), \quad (2.38)$$

$$s c^{R(2)} = -\theta^{kl} \partial_k c \partial_l \mathbb{C}^{(1)}, \quad (2.39)$$

and a part  $(a_i^{I(2)}, c^{I(2)})$  which only depends on the particular solutions  $(a_i^{(1)}, c^{(1)})$  of the first level,

$$sa_i^{I(2)} = \partial_i c^{I(2)} - \theta^{kl} \left( \partial_k c^{(1)} \partial_l A_i + \partial_k c \partial_l a_i^{(1)} \right), \quad (2.40)$$

$$sc^{I(2)} = -\theta^{kl} \partial_k c \partial_l c^{(1)}. \quad (2.41)$$

Different solutions of this last system have already been found in the literature [9–11]. We reproduce that of [9] for  $a_i^{I(2)}$ :

$$a_i^{I(2)} = \frac{1}{2} \theta^{kl} \theta^{mn} A_k (\partial_l A_m \partial_n A_i - \partial_l F_{mi} A_n + F_{lm} F_{ni}). \quad (2.42)$$

But all these particular solutions that we cited are connected by trivial cocycles of the form  $\partial_i Z^{(1)}$  [11], showing that they all belong to the same (trivial) cohomological class. Then, the work to be done is to solve the system (2.38). We first notice that if it was not for the presence of the trivial cocycle  $\mathbb{C}^{(1)}$ , the first equation of (2.38) would have exactly the form of equation (2.9) for  $n = 1$  with  $\mathbb{A}_i^{(1)}$  placed instead of  $A_i$ . So, our guess for the particular solution of (2.38) is to take  $a_i^{(1)}$  and  $c^{(1)}$  and change  $A_i$  for  $\mathbb{A}_i^{(1)}$  in the following form

$$a_i^{R(2)} = -\theta^{kl} \left( \mathbb{A}_k^{(1)} \partial_l A_i + A_k \partial_l \mathbb{A}_i^{(1)} - \frac{1}{2} \mathbb{A}_k^{(1)} \partial_i A_l - \frac{1}{2} A_k \partial_i \mathbb{A}_l^{(1)} \right), \quad (2.43)$$

$$c^{R(2)} = \frac{1}{2} \theta^{kl} (\partial_k c) \mathbb{A}_l^{(1)} + \frac{1}{2} \theta^{kl} \partial_k \mathbb{C}^{(1)} A_l.$$

In fact, as can be inferred from equations (2.7) and (2.8), the existence of a solution  $\mathbb{A}_i^{(n)}$ , eq. (2.33), at any given order  $n$  develops a problem analogous to that of the first order upon the next order  $n + 1$ , *i.e.*

$$sa_i^{R(n+1)} = \partial_i c^{R(n+1)} - \theta^{kl} \left( \partial_k \mathbb{C}^{(n)} \partial_l A_i + \partial_k c \partial_l \mathbb{A}_i^{(n)} \right), \quad (2.44)$$

$$sc^{R(n+1)} = -\theta^{kl} \partial_k c \partial_l \mathbb{C}^{(n)}, \quad (2.45)$$

which will have the same kind of solution as (2.43)

$$a_i^{R(n+1)} = -\theta^{kl} \left( \mathbb{A}_k^{(n)} \partial_l A_i + A_k \partial_l \mathbb{A}_i^{(n)} - \frac{1}{2} \mathbb{A}_k^{(n)} \partial_i A_l - \frac{1}{2} A_k \partial_i \mathbb{A}_l^{(n)} \right), \quad (2.46)$$

$$c^{R(n+1)} = \frac{1}{2} \theta^{kl} (\partial_k c) \mathbb{A}_l^{(n)} + \frac{1}{2} \theta^{kl} \partial_k \mathbb{C}^{(n)} A_l.$$

We can rewrite the expressions (2.43) in a compact way, making it clear the implicit redefinition of  $A_i$  and of  $c$ :

$$a_i^{R(2)} = a_i^{(1)} \left( A_i + \mathbb{A}_i^{(1)} \right) |_{\theta^2}, \quad (2.47)$$

$$c^{R(2)} = c^{(1)} \left( c + \mathbb{C}^{(1)}, A_i + \mathbb{A}_i^{(1)} \right) |_{\theta^2},$$

where  $|_{\theta^2}$  means projection on the  $\theta^2$  dependence of the polynomial. Obviously, this procedure can be continued to higher orders. For example, it is not difficult to show that the

effect of  $\mathbb{A}_i^{(1)}$  on the third order particular solution of the SW map is  $a_i^{(2)} \left( A_i + \mathbb{A}_i^{(1)} \right) |_{\theta^3}$ . And, as eqs. (2.46) show, each new element  $\mathbb{A}_i^{(n)}$  (2.33) of the cohomology of  $s$  modulo derivatives adjoined at each order in  $\theta$  means a new redefinition of the gauge connection. Recalling that the elements of  $\mathcal{H}^{(n)}(s)$  are gauge invariant polynomials constructed with curvatures and their derivatives, we indeed have shown the identification between the BRST sense of the SW map ambiguity [7, 5], given by  $\mathcal{H}^{(n)}(s)$ , and the well known freedom of the SW map by redefinitions of the commutative gauge potential [1, 3]. The BRST trivial part of the solution (2.33) for  $\mathbb{A}_i^{(n)}$ , being given by total derivatives  $\partial_i Z^{(n)}$ , is easily seen as the freedom of the SW map by field dependent gauge transformations [1, 3]

Finally, as there are no other possible elements allowed by the general solution (2.33), we answer the first question we proposed at the beginning of this section by saying that there is no room for other ambiguities in the SW map than these listed above.

It is time now to call attention to the different roles played by the cohomologically trivial and non-trivial parts of the general solution of the SW map. The trivial parts, as we just mentioned, have the form of gauge transformations. So, if we sum a BRST trivial term to any particular solution of the SW map, the final commutative gauge invariant action so mapped will not change at all. But a nontrivial solution of  $\mathcal{H}^{(n)}(s)$  in the cohomology of  $s/d$  will be able to modify any commutative action (see [12] for a study of the effects of gauge invariant field redefinitions on gauge invariant actions). After the understanding of this point, it becomes clear that there can be no uniquely defined commutative action coming from the SW map of a noncommutative theory if  $s/d$  nontrivial elements of  $\mathcal{H}^{(n)}(s)$  are found. This is a first hint in the way to answer the second question at the beginning of this section about the implications of the ambiguities on the mapping of noncommutative actions. Anyway, in the next sections we will extract some general information on the commutative theories coming from the SW map.

### 3. The general SW map for NCCS

From the developments of the last section, we saw that eqs. (1.9) and (1.10) are not a complete solution to the SW map problem. Already at first order there is a contribution to the solution in eq. (1.9) coming from elements of  $\mathcal{H}(s)$  which will be relevant to the SW mapping of noncommutative actions

$$\hat{A}_i(A) = A_i - \theta^{kl} (A_k \partial_l A_i - \frac{1}{2} A_k \partial_i A_l) + \mathbb{A}_i^{(1)} + O(\theta^2). \tag{3.1}$$

We also saw how the presence of nontrivial contributions of a given order  $n$  will alter the higher order terms, by generating a covariant mapping

$$A_i \rightarrow A_i + \mathbb{A}_i^{(n)} \tag{3.2}$$

in the particular solution of the SW map.

With this in mind, we are now in a position to analyse eqs. (1.11) and (1.12) and understand the consequences of its use. As described before, eq. (1.11) is a solution of a problem analogous to the first order SW map, and thus it is subjected to the same

limitations as those of the particular solution (1.9). In fact, equation (1.11) is only valid modulo field dependent gauge transformations and covariant field redefinitions (in [3] the authors have pointed out in this direction but incorporated only total derivatives (trivial contributions) in their analysis). This can also be noticed if one tries to write the second order expansion ( $\theta\delta_\theta$ ) of (1.11) using as first order solution eq. (1.9). Then, one finds that eq. (1.11) is not integrable at this order, if such “ambiguities” are not taken into account. In a certain sense, eq. particular solution (1.9) of the SW map. Thus, results coming from the use of eq. (1.11) will only refer to direct consequences of such a particular map and will not allow for the broader picture implied by the general solution (in the next section we will make use of the generalization of (1.11)).

This conclusion takes us back to eq. (1.13) developed in [4]. We said that there can be no uniquely defined commutative action coming from the SW map of a noncommutative theory when we have  $s/d$  nontrivial  $\mathcal{H}(s)$  contributions. But the conflict is now solved when we understand that eq. (1.13) comes as a direct consequence of eq. (1.11), and the latter is, by its turn, dependent on a particular solution of the SW map, the one associated to the  $s/d$  trivial solutions of  $\mathcal{H}(s)$ .

So, we will explicitly construct now a solution of the SW map with the intent of showing how the commutative Chern-Simons theory can be deformed by  $\theta$  terms coming from its noncommutative version.

In the first order in 3D, the only element of  $\mathcal{H}(s)$  is

$$\mathbb{A}_i^{(1)} = \alpha\theta^{kl}\partial_i F_{kl} \tag{3.3}$$

(just remembering that the original *NCCS* action does not contain the metric, so we do not include it on the field and parameter content of the theory). But this element becomes trivial in the  $s/d$  cohomology as it is a total derivative in the free index, and in this way it does not give any contribution to the commutative action.

In the second order, we can find an element of  $\mathcal{H}(s)$

$$\mathbb{A}_i^{(2)} = \alpha\theta^{ab}\theta^{ef}F_{ai}\partial_b F_{ef} \tag{3.4}$$

which cannot be written as a total derivative in the free index.

This suggests a deformation of the commutative *CS* action in the  $\theta^2$  order. In fact, writing the SW map up to second order as (compare with the usual solution of [9] in (2.42))

$$\begin{aligned} \hat{A}_i &= A_i - \frac{1}{2}\theta^{kl}(2A_k\partial_l A_i - A_k\partial_i A_L) \\ &+ \theta^{kl}\theta^{mn} \left[ \frac{1}{2}A_k((\partial_l A_m)\partial_n A_i - (\partial_l F_{mi})A_n + F_{lm}F_{ni}) + \alpha F_{ki}\partial_l F_{mn} \right] + o(\theta^3), \end{aligned} \tag{3.5}$$

we get a contribution from the SW map of the *NCCS* action in the commutative space beyond the usual *CS* term,

$$\begin{aligned} \hat{S}_{NCCS} &\longrightarrow S_{CS} + \frac{\alpha}{2} \int d^3x \epsilon^{abc}\theta^{kl}\theta^{mn} \times [F_{ka}\partial_l F_{mn}\partial_b A_c + A_a\partial_b(F_{kc}\partial_l F_{mn})] + o(\theta^3) \\ &= S_{CS} + \frac{\alpha}{2} \int d^3x \epsilon^{abc}\theta^{kl}\theta^{mn} [\partial_l(F_{ka}F_{mn})F_{bc}] + o(\theta^3). \end{aligned} \tag{3.6}$$

Such deformations of the  $CS$  theory we reach here were probably ignored so far because the solutions found for  $\hat{A}_i$  have consisted of cohomologically trivial variations (gauge transformations) of the same particular solution of the SW map [11]. Nevertheless we have to remark that the possible existence of nontrivial contributions had already been anticipated in [3].

The main point in the  $\theta$  deformations in the commutative space obtained in (3.6) is that they are solely expressed in terms of the curvature and derivatives. This indeed is not a specific feature of the order that we have analysed. As we have shown in section 2, the elements  $\mathbb{A}_i^{(n)} \in \mathcal{H}(s)$  generate a covariant mapping, eq. (3.2), in the SW map. Eq. (3.6) is an example of this fact, as it is just a mapping of the  $CS$  action by  $A_i \rightarrow A_i + \mathbb{A}_i^{(2)}$  up to second order. So, we can say that the general form of the commutative action after a SW map of the  $NCCS$  taking into account all possible  $\mathbb{A}_i^{(n)}$  is

$$\hat{S}_{NCCS} \longrightarrow S_{CS} + \frac{1}{2} \sum_{n=1}^{\infty} \int d^3x \epsilon^{abc} \mathbb{A}_a^{(n)} \left[ F_{bc} + \sum_{m=1}^{\infty} \partial_b \mathbb{A}_c^{(m)} \right], \quad (3.7)$$

and we see that the deformations of the  $CS$  action are all given by monomials constructed with the curvature and its derivatives. We can also assure that they all are interaction terms. The only possible contribution to a kinetic term would come from the first order (3.3). But this first order term, being a pure gauge, does not deform the  $CS$  action.

In [13] actions as (3.7) were studied. It was then shown that even these non-power-counting interactions cannot change the topological character of the  $CS$  theory, at least, perturbatively. The sensible point is that the kinetic topological action induces a definition of the physical observables of the theory as link invariants, and these are not perturbed by the interaction terms (it is straightforward to generalize the argument in [13] to interactions with external parameters  $\theta$ ).

We thus conclude that, in spite of the deformations appearing in the action (3.7), the SW map of the  $NCCS$  leads to commutative actions physically equivalent to the 3D Chern-Simons theory from the perturbative point of view. This analysis, in this sense, complements the result of [4]. This reasoning is also in agreement with the result of [14], where the authors showed that the tree level  $CS$  coefficient is not renormalized when  $NCCS$  is quantized in the axial gauge.

#### 4. The general SW map for NCMCS

Let us turn now to another example in 3D theories. The noncommutative Maxwell-Chern-Simons model ( $NCMCS$ ) has also been extensively studied [15–19]. The  $NCMCS$  action is

$$\hat{S}_{NCMCS} = \int d^3x \left[ -\frac{1}{4} \hat{F}_{ij} * \hat{F}^{ij} + \frac{m}{2} \epsilon^{ijk} \left( \hat{A}_i \partial_j \hat{A}_k - \frac{2i}{3} \hat{A}_i (\hat{A}_j * \hat{A}_k) \right) \right], \quad (4.1)$$

where  $m$  is the 3D noncommutative topological mass. The particular solution of the SW map leads to a complicated non-power-counting commutative action, which has already

been calculated up to the second order in  $\theta$  [17]

$$\begin{aligned} \hat{S}_{NCMCS} \xrightarrow{SW\text{map}} S_{MCS} - \frac{1}{2}\theta^{ij} \int d^3x \left( F_{jk} F^{kl} F_{li} - \frac{1}{4} F_{ij} F^{kl} F_{kl} \right) \\ + \frac{1}{4}\theta^{ij}\theta^{kl} \int d^3x \left( 2F_{jk} F_{lm} F^{mn} F_{ni} + F_{jm} F_k^m F_l^n F_{ni} - F_{ij} F_{km} F^{mn} F_{nl} + \right. \\ \left. - \frac{1}{8} F_{ij} F_{kl} F_{mn} F^{mn} + \frac{1}{4} F_{jk} F_{li} F_{mn} F^{mn} \right) + O(\theta^3). \end{aligned} \quad (4.2)$$

Obviously, all these interaction terms come from the mapping of the noncommutative Maxwell term in (4.1), as the  $NCCS$  term is not transformed by the particular  $SW$  map of (1.9) [4]. In [20] an argument was given for concluding that the interaction terms will always depend only on the field strength  $F$  at any order in  $\theta$ . This was latter formally proved in [21] so that

$$\hat{S}_{NCMCS} \longrightarrow S_{MCS} + \mathcal{L}(\theta, F). \quad (4.3)$$

In the previous section, we showed how the covariant contributions to the  $SW$  map change the form of the commutative action coming from the  $NCCS$  action. We can ask here, in the  $NCMCS$  case, what role the covariant contributions can play.

The authors in [19] showed that a term of the form  $\frac{1}{m}\epsilon_{ijk}\theta^{jk}F^2$  could be used to cancel part of the first order in  $\theta$  term in (4.2), although other first order terms would then be generated. This prompts us to generalize the idea a little bit further and find the complete form of the covariant term that should be added to (1.9) in order to cancel the first order terms in (4.2). Indeed we found that

$$\mathbb{A}_i^{(1)} = \frac{1}{2m}\epsilon_{ijk}\theta^{lj}F^{kn}F_{nl} - \frac{1}{8m}\epsilon_{ijk}\theta^{jk}F^2, \quad (4.4)$$

when substituted in the  $NCCS$  sector of (4.1), is able to cancel the first order contribution of (4.2). Notice that the terms in (4.4) for  $\mathbb{A}_i^{(1)}$  were not allowed in the pure  $CS$  case, eq. (3.3), since they depend on the metric, which was not part of the field and parameter content of the theory. But when we substitute (4.4) in the  $NCM$  sector of (4.1), we generate  $\theta$  first order terms in  $\frac{1}{m}$ . These can again be cancelled by covariant terms in  $\frac{1}{m^2}$  in the  $NCCS$  sector, and we see that the complete  $\mathbb{A}_i^{(1)}$  is in fact an infinite series in powers of  $\frac{1}{m}$ . The same process will happen in each  $\theta$  order of the covariant ambiguities  $\mathbb{A}_i^{(n)}$ .

The existence of such covariant mappings is not really a novelty. In [12], the authors proved that covariant mappings can always be defined in a way to reabsorb into the 3D free Maxwell-Chern-Simons action interaction terms constructed only with the field strength  $F$  and derivatives, which is just the case of (4.3). Further, as the covariant mappings are part of the ambiguities which are allowed in the  $SW$  map, all results of [12] can be adapted to this present case.

With this intent, we begin by writing the generalization of (1.11) taking into account the covariant ambiguities of the  $SW$  map,

$$\delta'_\theta \hat{A}_i = -\frac{1}{4}\delta\theta^{kl} \left\{ \hat{A}_k^* \partial_l \hat{A}_i + \hat{F}_{li} \right\} + \delta\theta^{kl} \hat{f}_{kli} \left( \hat{F}, \hat{D}, \theta \right) \quad (4.5)$$

where  $\hat{f}_{kli}$  stands for all possible covariant terms constructed with the curvatures  $\hat{F}$ , the Moyal covariant derivatives and the  $\theta$  parameters. A generalization of this kind was first proposed in [3], but only trivial terms (total derivatives) were considered for  $\hat{f}_{kli}$ . Obviously,  $\hat{f}_{kli}$  will transform covariantly in the noncommutative sense:

$$s \hat{f}_{kli} = i \left[ \hat{C}^*, \hat{f}_{kli} \right]. \quad (4.6)$$

Using eqs. (4.5) and (1.13), we get that after a general SW map the *NCCS* action can only depend on  $\theta$  through terms containing  $\hat{f}_{kli}$ ,

$$\delta'_\theta \hat{S}_{NCCS} = \frac{m}{2} \epsilon^{ijk} \delta\theta^{mn} \int d^3x \hat{f}_{mni} \hat{F}_{jk}. \quad (4.7)$$

But upon the *NCM* action, both parts of  $\delta'_\theta \hat{A}_i$  will contribute. First, we write the expression for  $\delta'_\theta \hat{F}_{kl}$ ,

$$\delta'_\theta \hat{F}_{kl} = \frac{\delta\theta^{mn}}{4} \left[ 2 \left\{ \hat{F}_{km}^*, \hat{F}_{ln} \right\} - \left\{ \hat{A}_m^*, \hat{D}_n \hat{F}_{kl} + \partial_n \hat{F}_{kl} \right\} + 4 \left( \hat{D}_k \hat{f}_{mnl} - \hat{D}_l \hat{f}_{mnk} \right) \right], \quad (4.8)$$

and then

$$\begin{aligned} \delta'_\theta \hat{S}_{NCM} = & -\frac{1}{2} \delta\theta^{mn} \int d^3x \left[ \left( \frac{1}{2} \left\{ \hat{F}_{im}^*, \hat{F}_{jn} \right\} - \frac{1}{4} \left\{ \hat{A}_m^*, \hat{D}_n \hat{F}_{ij} + \partial_n \hat{F}_{ij} \right\} \right) \hat{F}^{ij} \right] \\ & -\frac{1}{2} \delta\theta^{mn} \int d^3x \left[ \left( \hat{D}_i \hat{f}_{mnj} - \hat{D}_j \hat{f}_{mni} \right) \hat{F}^{ij} \right]. \end{aligned} \quad (4.9)$$

The sensible question is if it is possible to make

$$\delta'_\theta \hat{S}_{NCCS} + \delta'_\theta \hat{S}_{NCM} = 0 \quad (4.10)$$

with a convenient choice of  $\hat{f}_{ijk}$ . The first step is to write the second element in (4.9) exclusively in terms of  $\hat{F}_{ij}$ . This can be achieved as

$$\delta\theta^{mn} \int d^3x \left\{ \hat{A}_m^*, \hat{D}_n \hat{F}_{ij} + \partial_n \hat{F}_{ij} \right\} \hat{F}^{ij} = \frac{1}{2} \delta\theta^{mn} \int d^3x \left\{ \hat{F}_{mn}^*, \hat{F}_{ij} \right\} \hat{F}^{ij} \quad (4.11)$$

and (4.10) leads to the equation

$$\begin{aligned} \frac{m}{2} \epsilon^{ijk} \delta\theta^{mn} \int d^3x \hat{f}_{mni} \hat{F}_{jk} = & \frac{\delta\theta^{mn}}{2} \int d^3x \left( \frac{1}{2} \left\{ \hat{F}_{im}^*, \hat{F}_{jn} \right\} - \frac{1}{8} \left\{ \hat{F}_{mn}^*, \hat{F}_{ij} \right\} \right) \hat{F}^{ij} \\ & + \frac{\delta\theta^{mn}}{2} \int d^3x \left( 2 \hat{f}_{mni} \hat{D}_j \hat{F}^{ij} \right). \end{aligned} \quad (4.12)$$

This equation can be solved for  $\hat{f}_{ijk}$  as a series in powers of  $\frac{1}{m}$ ,

$$\hat{f}_{ijk} = \sum_{r=1}^{\infty} \hat{f}_{ijk}^{(r)}, \quad (4.13)$$

where the upper index  $r$  designates the power in  $\frac{1}{m}$ . The first term in the order  $\frac{1}{m}$  is

$$\hat{f}_{mni}^{(1)} = \frac{1}{4m} \epsilon_{nik} \left\{ \hat{F}_{jm}^* \hat{F}^{jk} \right\} - \frac{1}{16m} \epsilon_{mni} \left\{ \hat{F}_{jk}^* \hat{F}^{jk} \right\} \quad (4.14)$$

and the next orders can be obtained recursively from

$$\hat{f}_{mni}^{(r+1)} = -\frac{1}{m} \epsilon_{ijk} \hat{D}^k \hat{f}_{mn}^{(r)j}. \quad (4.15)$$

This means that with the choice of this specific covariant polynomial in (4.5) we can assure the independence of the noncommutative Maxwell-Chern-Simons theory on the  $\theta$  parameter, eq. (4.10), i.e.

$$\delta'_\theta \hat{S}_{\text{NCMCS}} = 0. \quad (4.16)$$

## 5. Conclusion

In this paper we studied the effects of the ambiguities of the Seiberg-Witten map on 3D noncommutative gauge theories. We showed how covariant ambiguities added to the normally used SW map deform the commutative Chern-Simons action by  $\theta$  interaction terms. We also showed how choosing adequate representatives of nontrivial (in the BRST sense) ambiguities the noncommutative Maxwell-Chern-Simons action can be mapped into the commutative version without the interaction terms which are usually associated to the SW mapping of Maxwell type actions.

Although these two results apparently point out to different directions, they are, in fact, the same development seen from different points of view. In both cases, what we have shown is that among all possible SW maps classified by BRST cohomology, we can find one element which cancels the  $\theta$  contributions. It thus erases the memory of these theories from their noncommutative origin. They are mapped into renormalizable, and in the abelian cases we treat, even free, commutative theories. The only difference between both cases lies in the need for a nontrivial BRST element in the *NCMCS* case, whereas the *NCCS* case requires that only trivial BRST terms should be added to the particular solution of the SW map in order to reach the commutative pure *CS* theory.

Indeed, both cases have more in common. As it was shown in [12], what allows for the reabsorption of interactions made of covariant elements (such as curvatures and their covariant derivatives) by non-linear local gauge field redefinitions is the presence of the Chern-Simons term in the action (in [22] the pure Maxwell theory is mapped to pure Chern-Simons theory but that mapping is not a series of local terms). This was crucial in finding the solution to eq. (4.12), which led to eq. (4.16). We believe that this is a general feature of noncommutative actions with Schwarz type topological sectors (we will be reporting soon on 4D noncommutative BF theories [23]). We also believe that these theories will be well-behaved upon quantization, as long as gauge fixing conditions and all quantization procedures can be translated from the commutative to the noncommutative space (this has been shown for the pure *NCCS* theory by explicit calculations in the noncommutative space [14]), although the renormalizability of pure *NCBF* models has been questioned



in [24]. Unfortunately, the same argument indicates that purely geometrical theories, as for instance the pure noncommutative Maxwell theory, seem to have unavoidably the companion of power counting nonrenormalizable  $\theta$  interactions in their commutative versions after the SW map. The already found nonrenormalizability of  $NCQED$  [25] is an evidence of this fact. The ambiguities of the SW map seem to be of no hope in these cases.

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